Adjustment theory

an introduction

Adjustment theory

an introduction

P.J.G. Teunissen

Delft University of Technology
Faculty of Civil Engineering and Geosciences
Department of Mathematical Geodesy and Positioning

VSSD

Series on Mathematical Geodesy and Positioning
[http://www.vssd.nl/hlf/a030.htm]

Adjustment Theory
P.J.G. Teunissen
2003 / viii + 193 p. / ISBN 90-407-1974-8

Dynamic Data Processing
P.J.G. Teunissen
2001 / x + 241 pp. / ISBN 90-407-1976-4

Global Positioning System (in Dutch)
G.J. Husti
2000 / 151 pp. / ISBN 90-407-1977-2

Hydrography
C.D. de Jong, G. Lachapelle, S. Skone, I. Elema
2003 / x + 353i pp. / ISBN 90-407-2359-1

Testing Theory
P.J.G. Teunissen
2000 / 147+viii pp. / ISBN 90-407-1975-6

First edition 2000-2006

Published by:
VSSD
Leeghwaterstraat 42, 2628 CA Delft, The Netherlands
tel. +31 15 278 2124, e-mail: dap@vssd.nl
internet: www.delftacademicpress.nl/a029.php

A collection of digital pictures and an elctronic version can be made available for lecturers who adopt this book. Please send a request by e-mail to dap@vssd.nl

ISBN 90-407-1974-8
NUR 930

Keywords: adjustment theory, geodesy

Preface

As in other physical sciences, empirical data are used in Geodesy and Earth Observation to make inferences so as to describe the physical reality. Many such problems involve the determination of a number of unknown parameters which bear a linear (or linearized) relationship to the set of data. In order to be able to check for errors or to reduce for the effect these errors have on the final result, the collected data often exceed the minimum necessary for a unique determination (redundant data). As a consequence of (measurement) uncertainty, redundant data are usually inconsistent in the sense that each sufficient subset yields different results from another subset. To obtain a unique solution, consistency needs to be restored by applying corrections to the data. This computational process of making the data consistent such that the unknown parameters can be determined uniquely, is referred to as *adjustment*. This book presents the material for an introductory course on Adjustment theory. The main goal of this course is to convey the knowledge necessary to perform an optimal adjustment of redundant data. Knowledge of linear algebra, statistics and calculus at the undergraduate level is required. Some prerequisites, repeatedly needed in the book, are summarized in the Appendices A-F.
Following the *Introduction*, the elementary adjustment principles of 'least-squares', unbiased minimum variance' and 'maximum likelihood' are discussed and compared in *Chapter 1*. These results are generalized to the multivariate case in chapters 2 and 3. In *Chapter 2* the model of observation equations is presented, while the model of condition equations is discussed in *Chapter 3*. Observation equations and condition equations are dual to each other in the sense that the first gives a parametric representation of the model, while the second gives an implicit representation. Observables that are either stochastically or functionally related to another set of observables are referred to as y^R-variates. Their adjustment is discussed in *Chapter 4*. In *Chapter 5* we consider mixed model representations and in *Chapter 6* the partitioned models. Models may be partitioned in different ways: partitioning of the observation equations, partitioning of the unknown parameters, partitioning of the condition equations, or all of the above. These different partitionings are discussed and their relevance is shown for various applications. Up tot this point all the material is presented on the basis of the assumption that the data are linearly related. In geodetic applications however, there are only a few cases where this assumption holds true. In the last chapter, *Chapter 7*, we therefore start extending the theory from linear to nonlinear situations. In particular an introduction to the linearization and iteration of nonlinear models is given.

Many colleagues of the Department of Mathematical Geodesy and Positioning whose assistance made the completion of this book possible are greatly acknowledged. C.C.J.M. Tiberius took care of the editing, while the typing was done by Mrs. J. van der Bijl and Mrs. M.P.M. Scholtes, and the drawings by Mr. A. Smit. Various lecturers have taught the book's material over the past years. In particular the feedback and valuable recommendations of the lecturers C.C.J.M. Tiberius, F. Kenselaar, G.J. Husti and H.M. de Heus are acknowledged.

P.J.G. Teunissen
April, 2003

Contents

Introduction

The *calculus of observations* (in Dutch: *waarnemingsrekening*) can be regarded as the part of mathematical statistics that deals with the results of measurement. Mathematical statistics is often characterized by saying that it deals with the making of decisions in uncertain situations. In first instance this does not seem to relate to measurements. After all, we perform measurements in order to obtain sharp, objective and unambiguous quantitative information, information that is both correct and precise, in other words accurate information. Ideally we would like to have 'mathematical certainty'. But mathematical certainty pertains to statements like:

$$\text{if } a=b \text{ and } b=c \text{ and } c=d\text{, then } a=d.$$

If a, b, c, and d are measured terrain distances however, a surveyor to be sure will first compare d with a before stating that $a=d$. Mathematics is a product of our mind, whereas with measurements one is dealing with humans, instruments, and other materialistic things and circumstances, which not always follow the laws of logic. From experience we know that measurements produce results that are only sharp to a certain level. There is a certain vagueness attached to them and sometimes even, the results are simply wrong and not usable.
In order to eliminate uncertainty, it seems obvious that one seeks to improve the instruments, to educate people better and to try to control the circumstances better. This has been successfully pursued throughout the ages, which becomes clear when one considers the very successful development of geodetic instruments since 1600. However, there are still various reasons why measurements will always remain uncertain to some level:

1. 'Mathematical certainty' relates to abstractions and not to physical, technical or social matters. A mathematical model can often give a useful description of things and relations known from our world of experience, but still there will always remain an essential difference between a model and the real world.
2. The circumstances of measurement can be controlled to some level in the laboratory, but in nature they are uncontrollable and only partly descriptive.
3. When using improved methods and instrumentation, one still, because of the things mentioned under 1 and 2, will have uncertainty in the results of measurement, be it at another level.
4. Measurements are done with a purpose. Dependent on that purpose, some uncertainty is quite acceptable. Since better instruments and methods usually cost more, one will have to ask oneself the question whether the reduction in uncertainty is worth the extra costs.

Summarizing, it is fundamentally and practically impossible to obtain absolute certainty from measurements, while at the same time this is also often not needed on practical grounds. Measurements are executed to obtain quantitative information, information which is accurate enough for a specific purpose and at the same time cheap enough for that purpose. It is because of this intrinsic uncertainty in measurements, that we have to consider the calculus of observations. The calculus of observations deals with:

1. The description and analysis of measurement processes.
2. Computational methods that take care, in some sense, of the uncertainty in measurements.
3. The description of the quality of measurement and of the results derived therefrom.
4. Guidelines for the design of measurement set-ups, so as to reach optimal working procedures.

The calculus of observations is essential for a geodesist, its meaning comparable to what mechanics means for the civil-engineer or mechanical-engineer. In order to shed some more light on the type of problems the calculus of observations deals with, we take some surveying examples as illustration. Some experience in surveying will for instance already lead to questions such as:

1. Repeating a measurement does usually not give the same answer. How to describe this phenomenon?
2. Results of measurement can be corrupted by systematic or gross errors. Can these errors be traced and if so, how?
3. Geometric figures (e.g. a triangle) are, because of 2, usually measured with *redundant* (*overtallig*) observations (e.g. instead of two, all three angles of a triangle are measured). These redundant observations will usually not obey the mathematical rules that hold true for these geometric figures (e.g. the three angular measurements will usually not sum up to π). An *adjustment* (*vereffening*) is therefore needed to obtain consistent results again. What is the best way to perform such an adjustment?
4. How to describe the effect of 1 on the final result? Is it possible that the final result is still corrupted with non-detected gross errors?

Once we are able to answer questions like these, we will be able to determine the required measurement set-up from the given desired quality of the end product. In this book we mainly will deal with points 1 and 3, the elementary aspects of adjustment theory.

Functional and stochastic model

In order to analyze scientific or practical problems and/or make them accessible for a quantitative treatment, one usually tries to rid the problem from its unnecessary frills and describes its essential aspects by means of notions from an applicable mathematical theory. This is referred to as the making of a *mathematical model*. On the one hand the model should give a proper description of the situation, while, on the other hand, the model should not be needlessly complicated and difficult to apply. The determination of a proper model can be considered the 'art' of the discipline. Well-tried and well-tested experience, new experience from experiments, intuition and creativity, they all play a role in formulating the model.
Here we will restrict ourselves to an example from surveying. Assume that we want to determine the relative position of terrain-'points', whereby their differences in height are of no interest to us. If the distances between the 'points' are not too large, we are allowed, as experience tells us, to describe the situation by projecting the 'points' onto a level surface (niveauvlak) and by treating this surface as if it was a plane. For the description of the relative position of the points, we are then allowed to make use of the two-dimensional Euclidean geometry (vlakke

meetkunde), the theory of which goes back some two thousand years. This theory came into being as an abstraction from how our world was seen and experienced. But for us the theory is readily available, thus making it possible to anticipate on its use, for instance by using sharp and clear markings for the terrain-'points' such that they can be treated as if they were mathematically defined points. In this example, the two-dimensional Euclidean geometry acts as our *functional model*. That is, notions and rules from this theory (such as the sine- and cosine rule) are used to describe the relative position of the terrain-'points'.

Although the two-dimensional Euclidean geometry serves its purpose well for many surveying tasks, it is pointed out that the gratuitously use of a 'standard' model can be full of risks. Just like the laws of nature (natuurwetten), a model can only be declared 'valid' on the basis of experiments. We will come back to this important issue in some of our other courses.

When one measures some of the geometric elements of the above mentioned set of points, it means that one assigns a number (the observation) according to certain rules (the measurement procedure) to for instance an angle α (a mathematical notion). From experience we know that a measurement of a certain quantity does not necessarily give the same answer when repeated. In order to describe this phenomenon, one could repeat the measurements a great many times (e.g. 100). And on its turn the experiment itself could also be repeated. If it then turns out that the histograms of each experiment are sufficiently alike, that is, their shapes are sufficiently similar and their locations do not differ too much, one can describe the measurement variability by means of a stochastic variable. In this way notions from mathematical statistics are used in order to describe the inherent variability of measurement processes. This is referred to as the *stochastic model*.

In practice not every measurement will be repeated of course. In that case one will have to rely on the extrapolation of past experience. When measuring, not only the measurement result as a number is of interest, but also additional information such as the type of instrument used, the measurement procedure followed, the observer, the weather, etc. Taken together this information is referred to as the *registration* (*registratie*). The purpose of the registration is to establish a link with past experience or experiments such that a justifiable choice for the stochastic model can be made.

In surveying and geodesy one often may assume as stochastic model that the results of measurement (the observations) of a certain measurement method can be described as being an independent sample drawn (aselecte trekking) from a normal (Gaussian) distribution with a certain standard deviation. The mathematical *expectation* (*verwachting*) of the normally distributed stochastic variable (the observable, in Dutch: de waarnemingsgrootheid), is then assumed to equal the unknown angle etc. From this it follows, if the three angles α, β and γ of a triangle are measured, that the expectations of the three angular stochastic variables obey the following relation of the functional model:

$$\alpha + \beta + \gamma = \pi$$

For the individual observations themselves this is usually not the case. A sketch of the relation between terrain situation, functional model, measurement, experiment and stochastic model is given in figure 0.1. The functional and stochastic model together from the mathematical model.

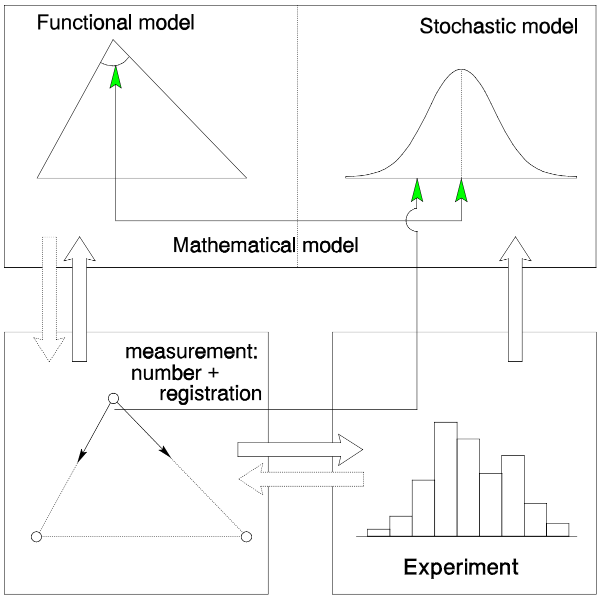

Figure 0.1: Diagram of the fundamental relations in adjustment theory