Dynamic data processing

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Foreword

This book is based on the lecture notes of the course *Dynamic data processing* as it has been given by the Department of Mathematical Geodesy and Positioning (MGP) of the Delft University of Technology since 1990. The prerequisites are a solid knowledge of adjustment theory and geodetic positioning, together with linear algebra, statistics and calculus. The theory and application of least-squares adjustment are treated in Adjustment theory (Delft University Press, 2000). The material of the present course extends the theory to the recursive estimation of time-varying or dynamic parameters. The time-varying parameters could for instance be geometric parameters such as position, attitude and shape, physical parameters such as temperature and humidity, or instrumental parameters such as clock drifts and biases. The time-varying parameters are said to be determined recursively when the method of determination enables sequential, rather than batch processing of the measurement data. The main goal is therefore to convey the knowledge necessary to be able to process sequentially collected measurement data in an optimal and efficient manner for the purpose of estimating time-varying parameters.

Following the *Introduction*, the basic theory of least-squares estimation is reviewed in *Chapter 1*. This is done for the model of observation equations and for the model of condition equations. In *Chapter 2* the principle of recursive least-squares estimation is introduced. The recursive principle allows one to update the least-squares solution for new observations without the need to store all past observations. Two different forms of the measurement-update equations are given. The results of Chapter 2, which hold true for time-invariant parameters, are generalized in *Chapter 3* to the case of time-varying parameters. The time-varying nature of the parameters is assumed captured by means of polynomial equations of motion. The recursive solution now consists of two types of update equations, the measurement-update equations and the time-update equations. Since there still exist many dynamic systems for which the rather simple polynomial model of Chapter 3 does not apply, a larger class of dynamic models is introduced in *Chapter 4*. These models are formulated using the state-space description of dynamic systems. In order to include randomness in the state-space description of dynamic systems, some of the elementary concepts of the theory of random functions are discussed in *Chapter 5*. This chapter also includes a description of the propagation laws for linear, time-varying systems. The results of Chapter 5 are used in *Chapter 6* to model possible uncertainties associated with the dynamic model. As a result the update equations are obtained for the recursive least-squares filtering and prediction of time-varying parameters.

Many colleagues of the Department of Mathematical Geodesy and Positioning whose assistance made the completion of this book possible are gratefully acknowledged. The typing of the book was done by Mrs. M.P.M. Scholtes, while C.D. de Jong took care of the editing. Various lecturers have taught the book's material over the past years. In particular the feedback and valuable recommendations of the lecturers H.M. de Heus, C.D. de Jong and C.C.J.M. Tiberius are acknowledged.

P.J.G. Teunissen
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Contents

Introduction

As in other physical sciences, empirical data are used in geodesy to make inferences so as to describe physical reality. Many such problems involve the determination of unknown parameters from a set of redundant measurements. Measurements are said to be redundant when they exceed the minimum necessary for a unique determination of the parameters. There are two main reasons for collecting redundant measurements. First the requirement to be able to check for mistakes or errors. Second the wish to increase the accuracy of the results computed. As a consequence of measurement uncertainty (exact measurements do not exist), the redundant data are usually inconsistent in the sense that each sufficient subset yields results which will differ from the results obtained from another subset. To obtain a unique solution, consistency needs to be restored by applying corrections to the data. This computational process of making the measurement data consistent with the model such that the unknown parameters can be determined uniquely, is referred to as adjustment. Adjustment theory therefore deals with the optimal combination of redundant measurements together with the estimation of unknown parameters. An introductory course on adjustment was presented in *Adjustment theory* (Delft University Press, 2000). This theory is extended in this book to the case of *time-varying* or dynamic parameters with an emphasis on their recursive estimation.

Time-varying parameters occur in many geodetic models. They could be geometric parameters such as position, attitude and shape, physical parameters such as temperature and humidity, or instrumental parameters such as clock drifts and biases. When a body (e.g. satellite, aircraft, car, or ship) is in motion, its position changes as function of time. Being able to track the position of such a moving object is of importance, for instance for navigation and guidance. A moving body may also change its attitude as function of time. Attitude determination is sometimes needed as an aid to navigation and guidance, but it also applies, in case of earth rotation, to the Earth as a whole. Objects that are subject to deformation change their shape as a function of time. On a global scale, for instance, the earth deforms due to various geophysical processes. But the earth's surface may also change its shape on more local or regional scales. Subsidence due to gas extraction is one such example. Apart from time-varying geometric parameters, also physical and instrumental parameters may change as function of time. Atmospheric parameters such as those of the ionosphere and troposphere, change on an hourly, daily and even seasonal basis. Also the performance of instruments often displays a dependence on time. This is the reason why calibrations are carried out, so as to keep the time-varying instrumental parameters in control.

A parameter solution is said to be recursive when the method of determination enables sequential, rather than batch processing of the measurement data. The need for a recursive solution is usually driven by the efficiency with which such solutions can be computed. This holds true in particular for applications in which the time-varying parameters need to be determined instantly or in real-time. We speak of a (near) real-time determination when the time of determination (almost) coincides with the time the parameter takes on the value to be determined. Such applications can typically be found in the area of navigation and guidance. In the case of navigation, for instance, it does not make sense to determine one's position with a too long time delay. In these applications there is therefore a real need to have a computational cycle time of the position determination that is as short as possible. This is feasible when recursive methods are used. But even in case real-time solutions are not an important issue, the use of recursive methods can still be attractive due to their computational efficiency.

When determining time-varying parameters from sequentially collected measurement data, one can discriminate between three types of estimation problems (see Figure 0.1). When the time at which a parameter estimate is required coincides with the time the last measurements are collected, the problem is referred to as *filtering*. When the time of interest falls within the time span of available measurement data, the problem is referred to as *smoothing*, and when the time of interest occurs after the time the last measurements are collected, the problem is called *prediction*. Thus filtering aims at the determination of current parameter values, while smoothing and prediction aim respectively at the determination of past and future parameter values. The emphasis in this book will be on recursive filtering.

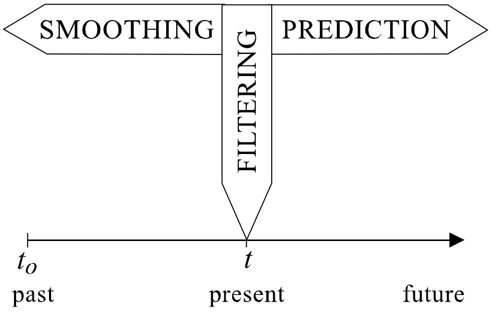

Figure 0.1: Prediction, filtering and smoothing.

The essence of a recursive method is that it enables one to update the parameter estimates for new measurements without the need to store all past measurements. Assume, for example, that one has collected at epoch *t-1* a redundant set of measurements y_{t-1} which bears a linear relationship with an unknown parameter vector x. The measurements y_{t-1} can then be used to obtain a linear least-squares estimate $\hat{x}_{t-1}$ of the unknown parameter vector x. Now assume that at the next epoch t a new set of measurements y_t becomes available which also bears a linear relationship to the same unknown parameter vector x. Since these additional measurements also contain information about the unknown parameter vector x, they can be used to improve the estimate $\hat{x}_{t-1}$ of x. One approach would be to use both y_{t-1} and y_t and to repeat the least-squares adjustment. As a result one obtains the improved least-squares estimate $\hat{x}_t$ of x. Although this approach is valid, it requires that one saves the past measurements y_{t-1}. In some cases this may be a too heavy computational burden, in particular if there are many past measurements or many epochs that precede the current epoch. Fortunately there is an alternative approach available, the recursive solution. It can be shown (under some mild restrictions) that the same improved least-squares estimate $\hat{x}_t$ of x, can also be computed from $\hat{x}_{t-1}$ and y_t instead of from y_{t-1} and y_t. The solution will then have the recursive structure:

$$\hat{x}_t = \hat{x}_{t-1} + K_t(y_t - A_t\hat{x}_{t-1})$$

in which K_t and A_t are matrices. This recursive equation, which holds true for any epoch t, is referred to as the *measurement-update* equation: the new measurements y_t are used to update the previous parameter estimate $\hat{x}_{t-1}$ so as to obtain the current parameter estimate $\hat{x}_t$.

Some elements of recursive estimation were already briefly introduced in *Adjustment theory* (Chapter 6, Section 3). However, just as in the above example, this brief introduction only dealt with models in which the parameter vector remained constant in time. In this book we will extend the theory to the case of time-varying parameters. This implies that some additional modeling needs to be done, namely one that describes the time-dependence of the parameter vector. Depending on the application at hand, these equations of motion can be of a kinematic or of a dynamic nature. Kinematics is used to relate position, velocity, acceleration and time without reference to the cause of motion, whereas dynamics also includes an explicit description of the forces responsible for the motion. As a consequence of having incorporated the time-varying nature of the parameter vector into the model, the recursion will now consist of two different update equations, the *time-update* (TU) and the *measurement update* (MU):

$$\hat{x}_{t|t-1} = \Phi_{t,t-1}\hat{x}_{t-1|t-1} \text{ (TU) and } \hat{x}_{t|t} = \hat{x}_{t|t-1} + K_t(y_t - A_t\hat{x}_{t|t-1}) \text{ (MU)}$$

with $\Phi_{t,t-1}$ the transition matrix. The time-update uses the filtered estimate $\hat{x}_{t-1|t-1}$ of epoch *t-1* to predict the parameter vector of the next epoch, x_t, as $\hat{x}_{t|t-1}$. This predicted estimate together with the new measurements y_t are then combined in the measurement update to obtain the filtered estimate of x_t as $\hat{x}_{t|t}$.

1 Least-squares: a review

1.1 The linear A-model

1.1.1 Consistency and inconsistency

Assume that we want to determine n parameters $x_\alpha \in \mathbb{R}$, $\alpha = 1,...,n$. An m-number of measurements $y_i \in \mathbb{R}$, $i = 1,...,m$, are carried out to determine these parameters. If the measurements bear a known linear relationship with the unknown parameters, we may write the model of observation equations as:

(1)
$$y_i = \sum_{\alpha=1}^{n} a_{i\alpha} x_\alpha \quad , \quad i = 1,...,m .$$

In this equation the known scalars $a_{i\alpha}$ model the assumed linear relationships between the measurements y_i and the parameters x_α. By introducing the matrix and vectors:

$$\underset{m\times n}{A} = \begin{pmatrix} a_{11} & \cdots & a_{1n} \\ \vdots & & \vdots \\ a_{m1} & \cdots & a_{mn} \end{pmatrix} \quad \underset{m\times 1}{y} = \begin{pmatrix} y_1 \\ \vdots \\ y_m \end{pmatrix} \quad \underset{n\times 1}{x} = \begin{pmatrix} x_1 \\ \vdots \\ x_n \end{pmatrix}$$

equation (1) can be written in matrix-vector form as:

(2)
$$\underset{m\times 1}{y} = \underset{m\times n}{A} \; \underset{n\times 1}{x} .$$

This is a system of an m-number of linear equations in an n-number of unknown parameters. It is now of interest to know under what conditions a solution to the linear system (2) exists and if a solution exists, whether it is unique or not. It will be clear that a solution to (2) exists if and only if the vector y can be written as a linear combination of the column vectors of matrix A. If this is the case the vector y is an element of the column space or range space of matrix A. This space is denoted as $R(A)$. Thus a solution to (2) exists if and only if:

(3)
$$y \in R(A) .$$

Systems for which this holds are called consistent systems. A system is said to be inconsistent if and only if:

(4)
$$y \notin R(A) .$$

In this case the vector y cannot be written as a linear combination of the column vectors of matrix A and hence no vector x exists such that (2) holds. The difference between consistency and inconsistency is depicted geometrically in Figure 1.1.