Numerical Methods
for
Partial Differential Equations

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Preface

This is a book about numerically solving partial differential equations occurring in technical and physical contexts and we (the authors) have set ourselves a more ambitious target than to just talk about the numerics. Our aim is to show the place of numerical solutions in the general modeling process and this must inevitably lead to considerations about modeling itself. Partial differential equations usually are a consequence of applying first principles to a technical or physical problem at hand. That means, that most of the time the physics also have to be taken into account especially for validation of the numerical solution obtained.

This book in other words is especially aimed at engineers and scientists who have 'real world' problems and it will concern itself less with pesky mathematical detail. For the interested reader though, we have included sections on mathematical theory to provide the necessary mathematical background.

This book is an abridged but improved version of our book [15]. The scope corresponds to Chapters 1-4, Section 9.7 and Chapters 10 and 11 from [15]. The material covers the FDM and FVM, but excludes the FEM, and is suitable for a semester course. The improvements will also be implemented in a future edition of the unabridged version [15] of this book.

Delft, August 2019

Jos van Kan
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Contents

Chapter 1

Review of some basic mathematical concepts

1.1 Preliminaries

In this chapter we take a bird's eye view of the contents of the book. Furthermore we establish a physical interpretation of certain mathematical notions, operators and theorems. As a first application we formulate a general conservation law, since conservation laws are the backbone of physical modeling. Finally we treat some mathematical theorems that will be used in the remainder of this book.

1.2 Global contents of the book

First, in Chapter 2, we take a look at second order partial differential equations and their relation with various physical problems. We distinguish between stationary (elliptic) problems and evolutionary (parabolic and hyperbolic) problems.

In Chapters 3 and 4 we look at numerical methods for elliptic equations. Chapter 3 deals with finite difference methods (FDM), of respectable age but still very much in use, while Chapter 4 is concerned with finite volume methods (FVM), a typical engineers option, constructed for conservation laws. In this special version of the book we do not discuss finite element methods (FEM), which have gained popularity over the last decades. These methods are discussed in the unabridged version [15] of the book, however.

Application of the FDM or FVM generally leaves us with a large set of algebraic equations. In Chapter 5 we focus on the difficulties that arise when these equations are nonlinear.

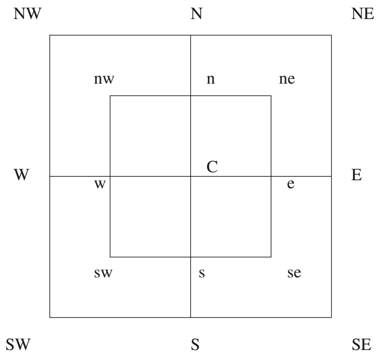

Figure 4.10: General control volume.

4.3.3 Boundary conditions

Boundary conditions of Dirichlet type do not present any problem, so we shall turn our attention to radiation boundary conditions of the form

$$\frac{\partial u}{\partial n} = \alpha(u_0 - u),$$

where we assume for simplicity that α and u_0 are constant. From an implementation point of view, it is easiest to take the nodal points *on* the boundary, which gives us a half cell control volume at the boundary like in Figure 4.11.

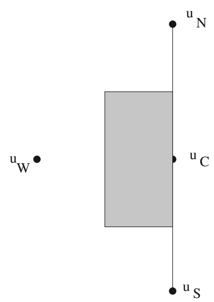

Figure 4.11: Boundary cell.

Integrating over the half volume and applying the divergence theorem we get:

$$-\left\{\frac{1}{r_C}\frac{u_S - u_C}{\Delta\theta}\frac{\Delta r}{2} + r_C\alpha(u_0 - u_C)\Delta\theta + \frac{1}{r_C}\frac{u_N - u_C}{\Delta\theta}\frac{\Delta r}{2} + r_w\frac{u_W - u_C}{\Delta r}\Delta\theta\right\} = f_C r_C\frac{\Delta r}{2}\Delta\theta, \quad (4.3.13)$$

where the radiation boundary condition has been substituted into the boundary integral of the right (east) boundary of the control volume.

Exercise 4.4.2 *Derive the discretization in the* displacement *variables u and v for Equation* (4.4.4b) *in the V_2 volume.* □

So apparently we must choose a grid in such a way that both V_1 and V_2 can be accommodated and the natural way to do that is take u and v in different nodal points, like in Figure 4.14.

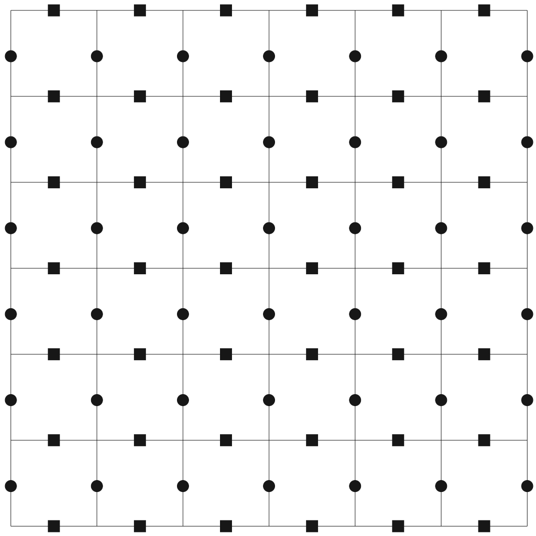

Figure 4.14: Staggered grid.

Such an arrangement of nodal point is called a *staggered grid*. This means that in general different problem variables reside in different nodes.

4.4.2 Boundary conditions

When discretizing a scalar equation you can often choose the grid in such a fashion that the boundary conditions can be easily implemented. With two or more components, especially on a staggered grid, this is no longer true.

Consider the W-boundary of our fixed plate in Figure 4.12. On this boundary we have the boundary conditions $u = 0$ and $v = 0$. A quick look at the staggered grid of Figure 4.14 shows a fly in the ointment. The u-points are on the boundary all right. Let us distinguish between equations derived from Equation (4.4.4a) (type 1) and those derived from Equation (4.4.4b) (type 2). In equations of type 1 you can easily implement the boundary conditions on the W-boundary. By the same token, you can easily implement the boundary condition on the N-boundary in type 2 equations. For equations of the ”wrong” type you have to resort to a trick. The generic form of an equation of type 2 in