

Numerical Methods
for
Partial Differential Equations

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Preface

This is a book about numerically solving partial differential equations occurring in technical and physical contexts and we (the authors) have set ourselves a more ambitious target than to just talk about the numerics. Our aim is to show the place of numerical solutions in the general modeling process and this must inevitably lead to considerations about modeling itself. Partial differential equations usually are a consequence of applying first principles to a technical or physical problem at hand. That means, that most of the time the physics also have to be taken into account especially for validation of the numerical solution obtained.

This book in other words is especially aimed at engineers and scientists who have 'real world' problems and it will concern itself less with pesky mathematical detail. For the interested reader though, we have included sections on mathematical theory to provide the necessary mathematical background.

This book is an abridged but improved version of our book [15]. The scope corresponds to Chapters 1-4, Section 9.7 and Chapters 10 and 11 from [15]. The material covers the FDM and FVM, but excludes the FEM, and is suitable for a semester course. The improvements will also be implemented in a future edition of the unabridged version [15] of this book.

Delft, August 2019

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Contents

1	Review of some basic mathematical concepts	1
1.1	Preliminaries	1
1.2	Global contents of the book	1
1.3	Building blocks for mathematical modeling	2
1.3.1	Gradient of a scalar	2
1.3.2	Directional derivative	4
1.3.3	Divergence of a vector field	5
1.3.4	Gauss' divergence theorem	6
1.3.5	Conservation laws	8
1.4	Preliminaries from linear algebra	9
1.5	The Poincaré inequality	14
1.6	Summary of Chapter 1	16
2	A crash course in PDEs	17
	Objectives	17
2.1	Classification	17
2.1.1	Three or more independent variables	19
2.2	Boundary and initial conditions	20
2.2.1	Boundary conditions	20
2.2.2	Initial conditions	22
2.3	Existence and uniqueness of a solution	22
2.3.1	The Laplace operator	22
2.3.2	The maximum principle and uniqueness	23
2.3.3	Existence	26
2.4	Examples	26
2.4.1	Flows driven by a potential	26
2.4.2	Convection-Diffusion	27
2.4.3	Navier-Stokes equations	27
2.4.4	Plane stress	29
2.4.5	Biharmonic equation	31
2.5	Summary of Chapter 2	32

3	Finite difference methods	33
	Objectives	33
3.1	The cable equation	33
3.1.1	Discretization	34
3.1.2	Properties of the discretization matrix A	36
3.1.3	Global error	38
3.2	Some simple extensions of the cable equation	40
3.2.1	Discretization of the diffusion equation	40
3.2.2	Boundary conditions	41
3.3	Singularly perturbed problems	44
3.3.1	Analytical solution	44
3.3.2	Numerical approximation	45
3.4	Poisson's equation on a rectangle	50
3.4.1	Matrix vector form	51
3.5	Boundary conditions extended	53
3.5.1	Natural boundary conditions	53
3.5.2	Dirichlet boundary conditions on non-rectangular regions	53
3.6	Global error estimate	55
3.6.1	The discrete maximum principle	55
3.6.2	Super solutions	58
3.7	Boundary fitted coordinates	60
3.8	Summary of Chapter 3	62
4	Finite volume methods	63
	Objectives	63
4.1	Heat transfer with varying coefficient	63
4.1.1	The boundaries	65
4.1.2	Conservation	66
4.1.3	Error in the temperatures	67
4.2	The stationary diffusion equation in 2 dimensions	68
4.2.1	Boundary conditions in case of a vertex-centered method	70
4.2.2	Boundary conditions in case of a cell-centered method	71
4.2.3	Boundary cells in case of a skewed boundary	73
4.2.4	Error considerations in the interior	74
4.2.5	Error considerations at the boundary	75
4.3	Laplacian in general coordinates	75
4.3.1	Transformation from Cartesian to General coordinates	75
4.3.2	An example of finite volumes in polar coordinates	77
4.3.3	Boundary conditions	79
4.4	Finite volumes on two component fields	80
4.4.1	Staggered grids	81
4.4.2	Boundary conditions	82
4.5	Stokes equations for incompressible flow	85
4.6	Summary of Chapter 4	87

5	Non-linear equations	89
	Objectives	89
5.1	Picard iteration	89
5.2	Newton's method in more dimensions	92
5.2.1	Starting values	94
5.3	Summary of Chapter 5	95
6	The heat- or diffusion equation	97
	Objectives	97
6.1	A fundamental inequality	97
6.2	Method of lines	100
6.2.1	One-dimensional examples	101
6.2.2	Two-dimensional example	103
6.3	Consistency of the spatial discretization	104
6.4	Time integration	106
6.5	Stability of the numerical integration	107
6.5.1	Gershgorin's disk theorem	109
6.5.2	Stability analysis of Von Neumann	112
6.6	The accuracy of the time integration	113
6.7	Conclusions for the method of lines	115
6.8	Special difference methods for the heat equation	115
6.8.1	The principle of the ADI method	115
6.8.2	Formal description of the ADI method	117
6.9	Summary of Chapter 6	119
7	The wave equation	121
	Objectives	121
7.1	A fundamental equality	121
7.2	The method of lines	124
7.2.1	The error in the solution of the system	124
7.3	Numerical time integration	127
7.4	Stability of the numerical integration	127
7.5	Total dissipation and dispersion	128
7.6	Direct time integration of the second order system	131
7.7	The CFL criterion	133
7.8	Summary of Chapter 7	136

Chapter 1

Review of some basic mathematical concepts

1.1 Preliminaries

In this chapter we take a bird's eye view of the contents of the book. Furthermore we establish a physical interpretation of certain mathematical notions, operators and theorems. As a first application we formulate a general conservation law, since conservation laws are the backbone of physical modeling. Finally we treat some mathematical theorems that will be used in the remainder of this book.

1.2 Global contents of the book

First, in Chapter 2, we take a look at second order partial differential equations and their relation with various physical problems. We distinguish between stationary (elliptic) problems and evolutionary (parabolic and hyperbolic) problems.

In Chapters 3 and 4 we look at numerical methods for elliptic equations. Chapter 3 deals with finite difference methods (FDM), of respectable age but still very much in use, while Chapter 4 is concerned with finite volume methods (FVM), a typical engineers option, constructed for conservation laws. In this special version of the book we do not discuss finite element methods (FEM), which have gained popularity over the last decades. These methods are discussed in the unabridged version [15] of the book, however.

Application of the FDM or FVM generally leaves us with a large set of algebraic equations. In Chapter 5 we focus on the difficulties that arise when these equations are nonlinear.

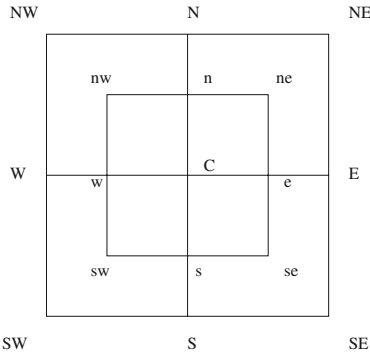


Figure 4.10: General control volume.

4.3.3 Boundary conditions

Boundary conditions of Dirichlet type do not present any problem, so we shall turn our attention to radiation boundary conditions of the form

$$\frac{\partial u}{\partial n} = \alpha(u_0 - u),$$

where we assume for simplicity that α and u_0 are constant. From an implementation point of view, it is easiest to take the nodal points *on* the boundary, which gives us a half cell control volume at the boundary like in Figure 4.11.

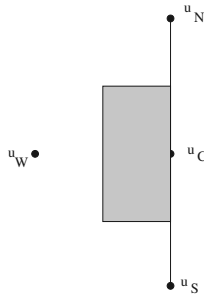


Figure 4.11: Boundary cell.

Integrating over the half volume and applying the divergence theorem we get:

$$-\left\{ \frac{1}{r_C} \frac{u_S - u_C}{\Delta\theta} \frac{\Delta r}{2} + r_C \alpha (u_0 - u_C) \Delta\theta + \frac{1}{r_C} \frac{u_N - u_C}{\Delta\theta} \frac{\Delta r}{2} + r_w \frac{u_W - u_C}{\Delta r} \Delta\theta \right\} = f_C r_C \frac{\Delta r}{2} \Delta\theta, \quad (4.3.13)$$

where the radiation boundary condition has been substituted into the boundary integral of the right (east) boundary of the control volume.

Exercise 4.4.2 Derive the discretization in the displacement variables u and v for Equation (4.4.4b) in the V_2 volume. \square

So apparently we must choose a grid in such a way that both V_1 and V_2 can be accommodated and the natural way to do that is take u and v in different nodal points, like in Figure 4.14.

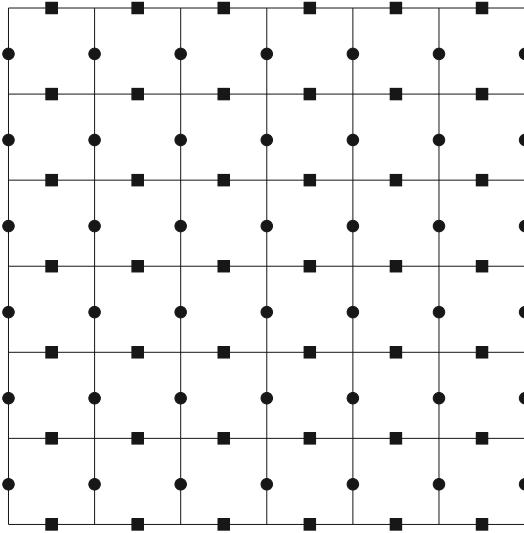


Figure 4.14: Staggered grid.

Such an arrangement of nodal point is called a *staggered grid*. This means that in general different problem variables reside in different nodes.

4.4.2 Boundary conditions

When discretizing a scalar equation you can often choose the grid in such a fashion that the boundary conditions can be easily implemented. With two or more components, especially on a staggered grid, this is no longer true.

Consider the W -boundary of our fixed plate in Figure 4.12. On this boundary we have the boundary conditions $u = 0$ and $v = 0$. A quick look at the staggered grid of Figure 4.14 shows a fly in the ointment. The u -points are on the boundary all right. Let us distinguish between equations derived from Equation (4.4.4a) (type 1) and those derived from Equation (4.4.4b) (type 2). In equations of type 1 you can easily implement the boundary conditions on the W -boundary. By the same token, you can easily implement the boundary condition on the N -boundary in type 2 equations. For equations of the "wrong" type you have to resort to a trick. The generic form of an equation of type 2 in