Electromagnetic Waves

— An Introductory Course

Martin Verweij (1961) graduated in 1992 from Delft University of Technology, Delft, Netherlands. From 1993-1997, he was a research fellow of the Royal Netherlands Academy of Arts and Sciences at the Laboratory of Electromagnetic Research of the Delft University of Technology, and since 1998 he is associate professor in the same laboratory. He has written a range of papers on integral transformation methods for electromagnetic, acoustic and elastic waves. He has been teaching various classes on basic and advanced electromagnetic wave theory since 1994 and was chosen Best Teacher of Electrical Engineering in 2003.

Peter van den Berg (1943) was a member of the scientific staff of the Laboratory of Electromagnetic Research, Delft University of Technology, Delft, Netherlands. He graduated in 1971 from the Delft University of Technology, and during the academic year 1973-1974 he was Visiting Lecturer in the Department of Mathematics, University of Dundee, Scotland. He was appointed as a full professor at the Laboratory of Electromagnetic Research in 1981. Since 2003 he is a research professor in the Faculty of Applied Sciences of the Delft University of Technology. During his career, he has written an impressive amount of papers on the numerical analysis of forward and inverse electromagnetic wavefield problems, and he has been teaching various classes on electromagnetic, acoustic and elastic waves.

Hans Blok (1935) was a member of the scientific staff of the Laboratory of Electromagnetic Research, Delft University of Technology, Delft, Netherlands. He graduated in 1970 from the Delft University of Technology, and he was appointed full professor at the Laboratory of Electromagnetic Research in 1980. He was dean of the Faculty of Electrical Engineering, Delft University of Technology in the period 1980-1982. During the academic year 1983-1984 he was a visiting scientist at Schlumberger-Doll Research, Ridgefield, Connecticut, U.S.A. During his career, he has written a number of papers on resonators and optical waveguides, and he has been teaching various classes on electromagnetic waves and signal theory. He has been emeritus professor since 2000..

Electromagnetic Waves

— An Introductory Course

dr. M.D. Verweij

prof.dr. P.M. van den Berg

prof.dr. H. Blok

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Preface

The course "Electromagnetic Waves" offers an introduction in the theoretical concepts of electromagnetic waves. This course book contains the basic material on time-varying wavefields and their applications in electrical engineering, e.g., electromagnetic compatibility, communication and remote sensing. A prerequisite to this course is a standard course "Electricity and Magnetism" where, from experimental laws, the Maxwell equations for time-varying electromagnetic fields are formulated as a system of partial differential equations.

Chapter 1 reviews the necessary mathematical background, while Chapter 2 introduces the fundamental mathematical equations: the Maxwell equations, the constitutive relations and boundary conditions. The main line of the course is the construction of solutions to these equations in some simple configurations. The concept of an electromagnetic wave is introduced in Chapter 3, where one-dimensional waves are discussed. A wave phenomenon can only be understood in connection with an electromagnetic source that generates a wave. For the excitation of one-dimensional waves, the planar-electric-current sheet is chosen. As a simple example of one-dimensional wave propagation, the parallel-plate waveguide is discussed shortly. In Chapter 4, the two-dimensional waves are studied, in particular specific properties as interference, Fresnel reflection/transmission factors, Brewster's angle and total reflection are treated. In Chapter 5, the consequences of a weakly inhomogeneous medium are discussed and the theory of electromagnetic rays is introduced. Further, in Chapters 6 and 7, the theory of transmission lines and electromagnetic waveguides is treated. In view of communication applications, the closed parallel-plate waveguide and the open dielectric-slab waveguide are described in full detail. Finally, Chapter 8 deals with the excitation of two-dimensional waves and the concept of the far-field approximation is introduced.

The student who has successfully completed the present introductory course on electromagnetic waves, has learned the basic concepts of electromagnetic wave propagation. By simplifying the problems in such a way that a description in terms of one-dimensional and two-dimensional waves suffices, more attention can be given to the physical understanding of the propagation phenomena. However, it is stressed that in more realistic configurations of present-day technology, a full three-dimensional description of electromagnetic wavefields is needed. In this context, it is noted that the methodology of handling the radiation and scattering of electromagnetic waves in three-dimensional configurations will be treated in more advanced courses of the electrical engineering curriculum.

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Delft, January 1999

H. Blok
P.M. van den Berg

Preface to the second edition

This edition is identical to the first edition, except that a number of errors have been corrected. The authors acknowledge their collegues of the Laboratory of Electromagnetic Research and in particular Dr. D. Quak and Mr. P. Jorna for reporting most of these errors.

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M.D. Verweij
P.M. van den Berg
H. Blok

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M.D. Verweij
P.M. van den Berg
H. Blok

Contents

Chapter 1

Introduction

Electromagnetic wavefields have a wide range of applications: from communication to medical treatment, from environmental sensing to energy radiation.

When following an electromagnetic wave on its course, we start with its excitation by an electromagnetic source. Some of them are natural sources such as the sun and stars, others are artificial ones (a transmitting antenna, a laser). Once it has been generated, the wave propagates along a certain path from the source to the receiver. Depending on the properties of the medium through which the wave passes, this propagation can lead to continuous refraction by spatial changes in the medium parameters (for example, the atmosphere), or to discontinuous refraction by an abrupt change in the medium (for example, an interface between two different media). Finally, the wave motion is picked up by an electromagnetic receiver (a receiving antenna, an optical detector).

Each of these aspects is the subject of theoretical and experimental investigation. Usually, when the attention is focussed on a particular detail, the remaining circumstances are chosen as simple as possible. For example, when one wants to investigate the directional characteristics of a transmitting antenna, the surrounding medium will be taken of the utmost simplicity, as far as its electromagnetic properties are concerned, and of infinite extent. When studying refraction phenomena during the propagation of an electromagnetic wave, the source will be taken a simple one, while the influence of the receiver will be neglected at all. These simplifications are dictated by the impossibility to take into account the influence of all parameters simultaneously.

The basic laws of macroscopic electromagnetic theory were formulated by James Clerk Maxwell and can be found in his famous book (MAXWELL 1873). For a survey of the history of the subject the reader is referred to Whittaker (WHITTAKER 1953). From the theory it follows that there exist electromagnetic waves that travel with a finite speed which in vacuo seems to be a universal constant, independent of the state of motion in which the observer carries out his or her experiments. (The latter is not the case for waves in matter.) Since through a wave motion with constant speed the changes in position in space and the changes in time are interrelated in a rigid manner, electromagnetic waves in vacuum can serve to interconnect the space-time observations for two observers in relative motion. This concept has led Einstein (EINSTEIN 1956) to the theory of relativity. We shall confine our analysis of electromagnetic waves to the case where the sources that generate the wavefield, and the observer are at rest with respect to the material media in the configuration.

As in any type of wave motion, the physical quantities that describe the electromagnetic waves, depend on position and time. Their time dependence in the domain where the source is acting is impressed by the excitation mechanism of the source. The subsequent dependence on position and time elsewhere is governed by propagation laws. The physical laws that underly the properties of waves are induced from a series of basic standard experiments. To carry out these experiments, an observer must be able to register the position and the instant at which an observation is made. To register the position the existence of an isotropic background space is preassumed. In this space, distance can be measured along three mutually perpendicular directions with one and the same position- and orientation-independent standard measuring rod. To register instants, the existence of a position- and orientation-independent standard clock is preassumed. The standard measuring rod is used to define, at a certain position which is denoted as the origin $\mathcal{O}$, an orthogonal Cartesian reference frame consisting of three base vectors $\{\boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ that are of unit length each. The orientations of these three base vectors form a mutually perpendicular, right-handed triad (Fig. 1.1). (The property that each base vector specifies geometrically a length and an orientation, makes it a vectorial quantity, or a vector; notationally, vectors will be represented by bold face symbols.) Let $\{x_1, x_2, x_3\}$ denote the three numbers that are needed to specify the position of an observer, then the vectorial position of the observer $\boldsymbol{x}$ is the linear combination (Fig. 1.2)

$$\boldsymbol{x} = x_1\boldsymbol{i}_1 + x_2\boldsymbol{i}_2 + x_3\boldsymbol{i}_3 \; . \tag{1.1}$$

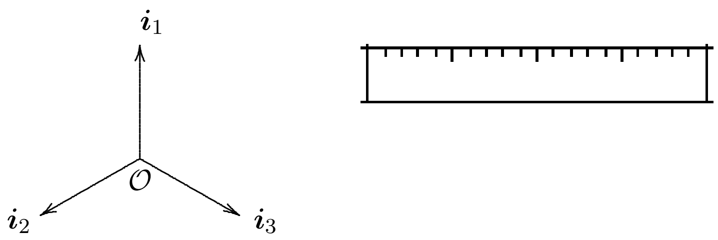

Figure 1.1. Standard measuring rod and Cartesian reference frame $\{O, \boldsymbol{i}_1, \boldsymbol{i}_2, \boldsymbol{i}_3\}$ in three-dimensional space.

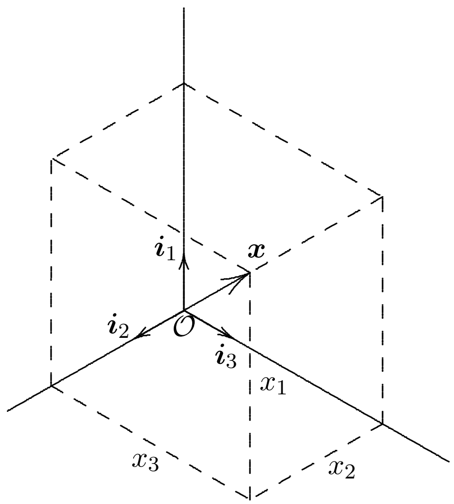

Figure 1.2. Cartesian coordinates $\{x_1, x_2, x_3\}$.

The numbers $\{x_1, x_2, x_3\}$ are denoted as the orthogonal Cartesian coordinates of the point of observation. The time coordinate is denoted by t.

One of the purposes of the basic standard experiments is to define the units in terms of which the measured physical quantities are expressed. In accordance with international convention, we employ the International System of Units (Système International d'Unités), abbreviated to SI, for expressing the measured physical quantities. The mathematical framework by which the results from the standard basic experiments are cast into the macroscopic physical laws that govern the wave motion is furnished by vector calculus. For this reason, the next section summarizes those properties of Cartesian vectors that are needed in our further analysis.

1.1. Cartesian vectors and their properties

The mathematical framework of the theory of electromagnetic waves is furnished by vector calculus. For this reason we summarize those properties of Cartesian vectors that are needed in our further analysis.

1.1.1. Addition, subtraction and multiplication of vectors

Vectors can be subjected to the algebraic operations of addition, subtraction and multiplication. Let the components of $\boldsymbol{v}$ be given by v_1, v_2 and v_3, and those of $\boldsymbol{w}$ by w_1, w_2 and w_3, then the components of the sum (difference) of $\boldsymbol{v}$ and $\boldsymbol{w}$ is given by

$$\boldsymbol{v} \pm \boldsymbol{w} = (v_1 \pm w_1)\boldsymbol{i}_1 + (v_2 \pm w_2)\boldsymbol{i}_2 + (v_3 \pm w_3)\boldsymbol{i}_3 \,. \tag{1.2}$$

The product of the scalar φ and the vector $\boldsymbol{v}$ is given by

$$\varphi\,\boldsymbol{v} = \varphi\,v_1\boldsymbol{i}_1 + \varphi\,v_2\boldsymbol{i}_2 + \varphi\,v_3\boldsymbol{i}_3 \,. \tag{1.3}$$

The scalar (dot) product of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is given by

$$\boldsymbol{v} \cdot \boldsymbol{w} = v_1 \cdot w_1 + v_2 \cdot w_2 + v_3 \cdot w_3 = \boldsymbol{w} \cdot \boldsymbol{v} \,, \tag{1.4}$$

The length of a vector $\boldsymbol{v}$ is denoted as

$$|\boldsymbol{v}| = (\boldsymbol{v} \cdot \boldsymbol{v})^{\frac{1}{2}} = (v_1^2 + v_2^2 + v_3^2)^{\frac{1}{2}} \,. \tag{1.5}$$

The vector (cross) product of the vectors $\boldsymbol{v}$ and $\boldsymbol{w}$ is given by

$$\boldsymbol{v} \times \boldsymbol{w} = (v_2 w_3 - v_3 w_2)\boldsymbol{i}_1 + (v_3 w_1 - v_1 w_3)\boldsymbol{i}_2 + (v_1 w_2 - v_2 w_1)\boldsymbol{i}_3 = -\boldsymbol{w} \times \boldsymbol{v} \,, \tag{1.6}$$

or in matrix notation

$$\boldsymbol{v} \times \boldsymbol{w} = \begin{vmatrix} \boldsymbol{i}_1 & \boldsymbol{i}_2 & \boldsymbol{i}_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \,. \tag{1.7}$$

The scalar triple product of three vectors $\boldsymbol{u}$, $\boldsymbol{v}$ and $\boldsymbol{w}$ is given by

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = u_1(v_2 w_3 - v_3 w_2) + u_2(v_3 w_1 - v_1 w_3) + u_3(v_1 w_2 - v_2 w_1) \,, \tag{1.8}$$

or in matrix notation

$$\boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = \begin{vmatrix} u_1 & u_2 & u_3 \\ v_1 & v_2 & v_3 \\ w_1 & w_2 & w_3 \end{vmatrix} \,. \tag{1.9}$$

The scalar triple product has the property

$$\begin{aligned} \boldsymbol{u} \cdot (\boldsymbol{v} \times \boldsymbol{w}) &= \boldsymbol{v} \cdot (\boldsymbol{w} \times \boldsymbol{u}) = \boldsymbol{w} \cdot (\boldsymbol{u} \times \boldsymbol{v}) \\ &= -\boldsymbol{w} \cdot (\boldsymbol{v} \times \boldsymbol{u}) = -\boldsymbol{v} \cdot (\boldsymbol{u} \times \boldsymbol{w}) = -\boldsymbol{u} \cdot (\boldsymbol{w} \times \boldsymbol{v}) \,. \end{aligned} \tag{1.10}$$

The vectorial triple product can be written as

$$\boldsymbol{u} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{u} \cdot \boldsymbol{w})\boldsymbol{v} - (\boldsymbol{u} \cdot \boldsymbol{v})\boldsymbol{w} \,. \tag{1.11}$$

As regards the differentiation of a vector, two cases have to be distinguished: differentiation with respect to a parameter, and differentiation with respect to the spatial (Cartesian) coordinates of the space in which the vector function is defined.

1.1.2. Differentiation with respect to a parameter

Let $\varphi = \varphi(t)$ a scalar function and assume that φ is a differentiable function of the parameter t (in electromagnetics often the time coordinate). Then, the derivative $\partial_t\varphi = \partial\varphi/\partial t$ is also a scalar function. Let $\boldsymbol{v} = \boldsymbol{v}(t)$ be a vector function and assume that $\boldsymbol{v}$ is a differentiable function of the parameter t. Let $v_1 = v_1(t)$, $v_2 = v_2(t)$, and $v_3 = v_3(t)$ denote the components of $\boldsymbol{v}$, then the derivative $\partial_t\boldsymbol{v}$ of $\boldsymbol{v}$ is the vector

$$\partial_t\boldsymbol{v} = (\partial_t v_1)\boldsymbol{i}_1 + (\partial_t v_2)\boldsymbol{i}_2 + (\partial_t v_3)\boldsymbol{i}_3 \,. \tag{1.12}$$

Let $\varphi = \varphi(t)$ be a differentiable scalar function of the parameter t and let $\boldsymbol{v} = \boldsymbol{v}(t)$ and $\boldsymbol{w} = \boldsymbol{w}(t)$ be differentiable vector functions of the parameter t, then we have the following differentiation rules:

$$\partial_t(\varphi\,\boldsymbol{v}) = (\partial_t\varphi)\boldsymbol{v} + \varphi\partial_t\boldsymbol{v} \,, \tag{1.13}$$

$$\partial_t(\boldsymbol{v}\times\boldsymbol{w}) = (\partial_t\boldsymbol{v})\times\boldsymbol{w} + \boldsymbol{v}\times\partial_t\boldsymbol{w} \,. \tag{1.14}$$

1.1.3. Differentiation with respect to the spatial coordinates

Let φ be a scalar function and assume that $\varphi = \varphi(\boldsymbol{x}) = \varphi(x_1, x_2, x_3)$ is a differentiable function of the spatial (Cartesian) coordinates x_1, x_2 and x_3. Then, the derivatives $\partial_1\varphi = \partial\varphi/\partial x_1$, $\partial_2\varphi = \partial\varphi/\partial x_2$ and $\partial_3\varphi = \partial\varphi/\partial x_3$ are also scalar functions. In this context, the gradient of $\varphi = \varphi(\boldsymbol{x})$ is introduced as

$$\operatorname{grad}\varphi = \boldsymbol{\nabla}\varphi = (\partial_1\varphi)\boldsymbol{i}_1 + (\partial_2\varphi)\boldsymbol{i}_2 + (\partial_3\varphi)\boldsymbol{i}_3 \,, \tag{1.15}$$

where

$$\boldsymbol{\nabla} = \boldsymbol{i}_1\partial_1 + \boldsymbol{i}_2\partial_2 + \boldsymbol{i}_3\partial_3 \tag{1.16}$$

is the operator of Hamilton, the so-called nabla operator or del operator. This operator is a vector and acts as a spatial differentiation with respect to the three spatial coordinates.

Let $\boldsymbol{v}$ be a vector function and assume that $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x}) = \boldsymbol{v}(x_1, x_2, x_3)$ is a differentiable function of the spatial (Cartesian) coordinates x_1, x_2 and x_3. The derivative $\partial_1\boldsymbol{v}$ is the vector

$$\partial_1\boldsymbol{v} = (\partial_1 v_1)\boldsymbol{i}_1 + (\partial_1 v_2)\boldsymbol{i}_2 + (\partial_1 v_3)\boldsymbol{i}_3 \,. \tag{1.17}$$

Similarly, we have

$$\partial_2 \boldsymbol{v} = (\partial_2 v_1)\boldsymbol{i}_1 + (\partial_2 v_2)\boldsymbol{i}_2 + (\partial_2 v_3)\boldsymbol{i}_3 \,, \tag{1.18}$$

$$\partial_3 \boldsymbol{v} = (\partial_3 v_1)\boldsymbol{i}_1 + (\partial_3 v_2)\boldsymbol{i}_2 + (\partial_3 v_3)\boldsymbol{i}_3 \,. \tag{1.19}$$

These three derivatives operating on the vector function $\boldsymbol{v}$ can be combined in the divergence operator, defined as

$$\operatorname{div} \boldsymbol{v} = \boldsymbol{\nabla} \cdot \boldsymbol{v} = \partial_1 v_1 + \partial_2 v_2 + \partial_3 v_3 \,, \tag{1.20}$$

and in the curl operator, defined as

$$\operatorname{curl} \boldsymbol{v} = \boldsymbol{\nabla} \times \boldsymbol{v} = (\partial_2 v_3 - \partial_3 v_2)\boldsymbol{i}_1 + (\partial_3 v_1 - \partial_1 v_3)\boldsymbol{i}_2 + (\partial_1 v_2 - \partial_2 v_1)\boldsymbol{i}_3 \,. \tag{1.21}$$

We note that $\boldsymbol{\nabla}$ is a vector operator satisfying two sets of rules:

- vector rules;
- partial differentiation rules, including differentiation of a product.

We now summarize the rules for the differentiation with respect to the spatial coordinates of the scalar functions $\varphi = \varphi(\boldsymbol{x})$ and $\psi = \psi(\boldsymbol{x})$, and of the vector functions $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$ and $\boldsymbol{w} = \boldsymbol{w}(\boldsymbol{x})$.

$$\boldsymbol{\nabla}(\varphi + \psi) = \boldsymbol{\nabla}\varphi + \boldsymbol{\nabla}\psi \,, \tag{1.22}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{\nabla} \cdot \boldsymbol{v} + \boldsymbol{\nabla} \cdot \boldsymbol{w} \,, \tag{1.23}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{v} + \boldsymbol{w}) = \boldsymbol{\nabla} \times \boldsymbol{v} + \boldsymbol{\nabla} \times \boldsymbol{w} \,, \tag{1.24}$$

$$\boldsymbol{\nabla}(\varphi\,\psi) = (\boldsymbol{\nabla}\varphi)\,\psi + \varphi\,\boldsymbol{\nabla}\psi \,, \tag{1.25}$$

$$\boldsymbol{\nabla} \cdot (\varphi\,\boldsymbol{v}) = (\boldsymbol{\nabla}\varphi) \cdot \boldsymbol{v} + \varphi\,\boldsymbol{\nabla} \cdot \boldsymbol{v} \,, \tag{1.26}$$

$$\boldsymbol{\nabla} \times (\varphi\,\boldsymbol{v}) = (\boldsymbol{\nabla}\varphi) \times \boldsymbol{v} + \varphi\,\boldsymbol{\nabla} \times \boldsymbol{v} \,, \tag{1.27}$$

$$\boldsymbol{\nabla} \cdot (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{\nabla} \times \boldsymbol{v}) \cdot \boldsymbol{w} - \boldsymbol{v} \cdot (\boldsymbol{\nabla} \times \boldsymbol{w}) \,, \tag{1.28}$$

$$\boldsymbol{\nabla} \times (\boldsymbol{v} \times \boldsymbol{w}) = (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} - \boldsymbol{w}\,\boldsymbol{\nabla} \cdot \boldsymbol{v} - (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{w} + \boldsymbol{v}\,\boldsymbol{\nabla} \cdot \boldsymbol{w} \,, \tag{1.29}$$

$$\boldsymbol{\nabla}(\boldsymbol{v} \cdot \boldsymbol{w}) = \boldsymbol{w} \times (\boldsymbol{\nabla} \times \boldsymbol{v}) + (\boldsymbol{w} \cdot \boldsymbol{\nabla})\boldsymbol{v} + \boldsymbol{v} \times (\boldsymbol{\nabla} \times \boldsymbol{w}) + (\boldsymbol{v} \cdot \boldsymbol{\nabla})\boldsymbol{w} \,. \tag{1.30}$$

We note that we have assumed that the functions φ, ψ, $\boldsymbol{v}$ and $\boldsymbol{w}$ are differentiable functions of the spatial coordinates. When we assume that $\varphi = \varphi(\boldsymbol{x})$

is also a twice differentiable function of $\boldsymbol{x}$, we have the rules:

$$\begin{aligned} \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\varphi) &= (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\varphi = (\partial_1^2+\partial_2^2+\partial_3^2)\varphi\,, && (1.31)\\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\varphi) &= \mathbf{0}\,, && (1.32)\\ \boldsymbol{\nabla}\cdot(\boldsymbol{\nabla}\times\boldsymbol{v}) &= 0\,, && (1.33)\\ \boldsymbol{\nabla}\times(\boldsymbol{\nabla}\times\boldsymbol{v}) &= \boldsymbol{\nabla}(\boldsymbol{\nabla}\cdot\boldsymbol{v})-(\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})\boldsymbol{v}\,. && (1.34)\end{aligned}$$

Subsequently, we present the rules for the spatial differentiation of a spatially dependent function $f=f(|\boldsymbol{x}|)$:

$$\begin{aligned} \boldsymbol{\nabla}|\boldsymbol{x}| &= \frac{\boldsymbol{x}}{|\boldsymbol{x}|}\,, && (1.35)\\ \boldsymbol{\nabla}|\boldsymbol{x}|^n &= n|\boldsymbol{x}|^{n-2}\boldsymbol{x}\,, && (1.36)\\ \boldsymbol{\nabla}f(|\boldsymbol{x}|) &= \partial f(|\boldsymbol{x}|)\,\frac{\boldsymbol{x}}{|\boldsymbol{x}|}\,, && (1.37)\end{aligned}$$

where ∂f is the derivative of f with respect to its argument. Further, we have:

$$\begin{aligned} \boldsymbol{\nabla}\cdot\boldsymbol{x} &= 3\,, && (1.38)\\ \boldsymbol{\nabla}\times\boldsymbol{x} &= \mathbf{0}\,, && (1.39)\\ (\boldsymbol{\nabla}\cdot\boldsymbol{\nabla})|\boldsymbol{x}|^n &= n(n+1)|\boldsymbol{x}|^{n-2}\,, && (1.40)\end{aligned}$$

and when $\boldsymbol{a}$ is a constant vector:

$$\begin{aligned} \boldsymbol{\nabla}(\boldsymbol{a}\cdot\boldsymbol{x}) &= \boldsymbol{a}\,, && (1.41)\\ (\boldsymbol{a}\cdot\boldsymbol{\nabla})\boldsymbol{x} &= \boldsymbol{a}\,, && (1.42)\\ (\boldsymbol{a}\times\boldsymbol{\nabla})\times\boldsymbol{x} &= -2\boldsymbol{a}\,. && (1.43)\end{aligned}$$

Interpretation of grad φ

We consider a continuously differentiable scalar function $\varphi=\varphi(\boldsymbol{x})$ and we take the dot product of its gradient $\boldsymbol{\nabla}\varphi$ (del φ) and an infinitesimal increment of length

$$\mathrm{d}\boldsymbol{x} = \mathrm{d}x_1\boldsymbol{i}_1 + \mathrm{d}x_2\boldsymbol{i}_2 + \mathrm{d}x_3\boldsymbol{i}_3\,. \qquad (1.44)$$

Thus we obtain

$$(\boldsymbol{\nabla}\varphi)\cdot\mathrm{d}\boldsymbol{x} = (\partial_1\varphi)\mathrm{d}x_1 + (\partial_2\varphi)\mathrm{d}x_2 + (\partial_3\varphi)\mathrm{d}x_3 = \mathrm{d}\varphi\,, \qquad (1.45)$$

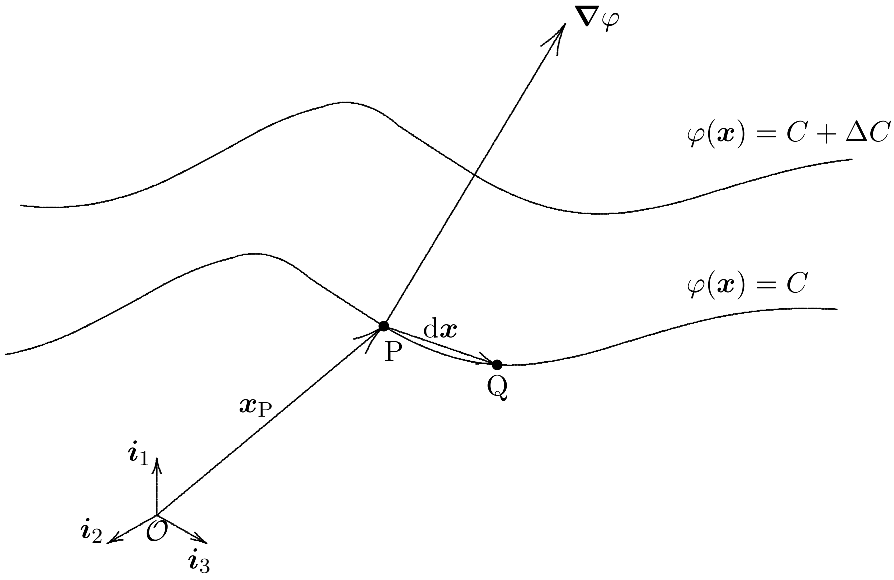

Figure 1.3. The gradient vector.

the change in the scalar function φ corresponding to a change in position $\mathrm{d}\boldsymbol{x}$. Now consider P and Q to be two points on a surface $\varphi(x_1, x_2, x_3) =$ constant. These points are chosen so that Q is a distance $\mathrm{d}\boldsymbol{x}$ from P. Then, moving from P to Q, the change in $\varphi(x_1, x_2, x_3) =$ constant is given by

$$\mathrm{d}\varphi = (\nabla\varphi) \cdot \mathrm{d}\boldsymbol{x} = 0\,, \tag{1.46}$$

since we stay on the surface $\varphi(x_1, x_2, x_3) =$ constant. This shows that $\nabla\varphi$ is perpendicular to $\mathrm{d}\boldsymbol{x}$. The vectorial distance $\mathrm{d}\boldsymbol{x}$ may have any direction from P as long as it stays in the surface $\varphi =$ constant, point Q being restricted to this surface. For vanishing $\mathrm{d}\boldsymbol{x}$, we observe that $\nabla\varphi$ is oriented in a direction of the normal to the surface $\varphi =$ constant (see Fig. 1.3).

If we now permit $\mathrm{d}\boldsymbol{x}$ to take us from one surface $\varphi = C$, C being a constant, to an adjacent surface $\varphi = C + \mathrm{d}C$, then,

$$\mathrm{d}\varphi = \mathrm{d}C = (\nabla\varphi) \cdot \mathrm{d}\boldsymbol{x}\,. \tag{1.47}$$

For a given $\mathrm{d}\varphi$, $|\mathrm{d}\boldsymbol{x}|$ is a minimum when it is chosen parallel to $\boldsymbol{\nabla}\varphi$, or, for a given $|\mathrm{d}\boldsymbol{x}|$, the change in the scalar function φ is maximized by choosing $\mathrm{d}\boldsymbol{x}$ parallel to $\boldsymbol{\nabla}\varphi$. This identifies $\boldsymbol{\nabla}\varphi$ as a vector having the direction of the maximum space rate of change of φ.

Very often the notion of *directional derivative* occurs. When $\boldsymbol{\tau}$ is a unit vector, the quantity $\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\varphi$ is called the directional derivative of φ in the direction of $\boldsymbol{\tau}$, and equals the rate of change of φ in the direction of $\boldsymbol{\tau}$, viz.,

$$\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\varphi = \partial_\tau\varphi = \tau_1\partial_1\varphi + \tau_2\partial_2\varphi + \tau_3\partial_3\varphi\,. \tag{1.48}$$

When $\boldsymbol{\tau}$ is the tangent along a surface $\varphi =$ constant, we obtain

$$\boldsymbol{\tau}\cdot\boldsymbol{\nabla}\varphi = \partial_\tau\varphi = 0\,, \tag{1.49}$$

which is consistent with Eq. (1.46).

Interpretation of $\operatorname{div}\boldsymbol{v}$

We consider a continuously differentiable vector function $\boldsymbol{v} = \boldsymbol{v}(\boldsymbol{x})$. The divergence operator $\boldsymbol{\nabla}\cdot\boldsymbol{v}$ (del dot $\boldsymbol{v}$) results in a scalar quantity indicating the outflow of a vector field. It can be obtained from the limiting behavior of the net outflow integral for a vanishing small enclosed volume. To show this we first compute the net outflow of a vector field $\boldsymbol{v}$ over the elementary domain with volume $\mathrm{dV} = \mathrm{d}x_1\mathrm{d}x_2\mathrm{d}x_3$ at the center of the elementary domain (see Fig. 1.4). This latter point is given by $\boldsymbol{x}_\mathrm{P} = \{\frac{1}{2}\mathrm{d}x_1, \frac{1}{2}\mathrm{d}x_2, \frac{1}{2}\mathrm{d}x_3\}$. By Taylor's theorem, the field component v_1 is

$$v_1(\boldsymbol{x}) = v_1(\boldsymbol{x}_\mathrm{P}) + (\partial_1 v_1)(x_1 - \tfrac{1}{2}\mathrm{d}x_1) + (\partial_2 v_1)(x_2 - \tfrac{1}{2}\mathrm{d}x_2) + (\partial_3 v_1)(x_3 - \tfrac{1}{2}\mathrm{d}x_3) + \text{higher order terms}\,. \tag{1.50}$$

The surface integral of the normal component of $\boldsymbol{v}$ (in the direction of the outward normal) over the top surface $\{x_1 = \mathrm{d}x_1,\ 0 < x_2 < \mathrm{d}x_2,\ 0 < x_3 < \mathrm{d}x_3\}$ of the volume element, shown in Fig. 1.4, is

$$\int_{x_2=0}^{\mathrm{d}x_2}\int_{x_3=0}^{\mathrm{d}x_3} v_1(\mathrm{d}x_1, x_2, x_3)\,\mathrm{dA} = [v_1(\boldsymbol{x}_\mathrm{P}) + \tfrac{1}{2}(\partial_1 v_1)\,\mathrm{d}x_1]\mathrm{d}x_2\mathrm{d}x_3 + \text{higher order terms}\,. \tag{1.51}$$